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ON SPINOR BUNDLES

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We find a method to obtain presentations of coherent sheaves on the Klein's quadric G, involving only line bundles and the two (elementary) spinor bundles: The universal rank-2 subbundle $\mathscr{E}'$ and the universal rank-2 quotient bundle $\mathscr{E}^*$. The theoretical meaning of our result is that all coherent sheaves on G are built out of these two bundles through certain exact sequences, although our method is practical only when the sheaf has a simple cohomology.

We first give a Beilison's [1] type spectral sequence (Theorem 1), then consider some auxiliary Leray spectral sequence in order to approach the problem of computing the annoying term involving $S^2\mathscr{E}'$. As an example of application we give a presentation of the spinor bundles $S^n\mathscr{E}$, $S^n\mathscr{E}'$ of current use in Physics (Penrose program [3]). The background in Commutative and Homological Algebra that we use can be found in [2] and [4].

This is to our knowledge the first example different of $\mathbb{P}^n$ where Beilinson's ideas [1] work, so raising the open problem of finding the most general context for the Beilinson's machine.

We first mention some elementary facts about Kleins quadric $G=\mathrm{Gr}(1,3)$ i.e. the Grassmann variety of lines in projective space $\mathbb{P}^3$, where K is an algebraically closed field. Its Chow ring is well known: $A^1(G)=\mathbb{Z}$. The generator here is the line bundle $\mathscr{O}_G(1)$ restriction of $\mathscr{O}_{\mathbb{P}^5}(1)$ when we see G as a four-dimensional quadric embedded in $\mathbb{P}^5: A^2(G)=\mathbb{Z}\oplus\mathbb{Z}$ and a cycle representing the generator $(0,1)$ is an α'-plane (β-plane in classic papers). The points of such a plane correspond to lines of $\mathbb{P}^3$ contained in a given plane. Cycles representing the $(1,0)$ generators are α-planes, i.e. planes whose points correspond to lines through a given point of $\mathbb{P}^3$.

We also recall that the elementary spinor bundles are the rank-2 vector bundles $\mathscr{E}, \mathscr{E}'$ on G appearing in the sequence

$$0\longrightarrow \mathscr{E}'\longrightarrow V\otimes\mathscr{O}_G\longrightarrow \mathscr{E}^*\longrightarrow 0$$

and its dual (where $V^*=H^0(\mathscr{O}_{\mathbb{P}^3}(1))$

$$0\longrightarrow \mathscr{E}\longrightarrow V^*\otimes\mathscr{O}_G\longrightarrow \mathscr{E}'^*\longrightarrow 0.$$

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They have Chern classes

$$c_1(\mathscr{E}) = -1, \qquad c_2(\mathscr{E}) = (1,0),$$
$$c_1(\mathscr{E}') = -1, \qquad c_2(\mathscr{E}') = (0,1).$$

Therefore $\mathscr{E}^* \cong \mathscr{E}(1)$, $\mathscr{E}'^* \cong \mathscr{E}'(1)$ and they have Chern classes

$$c_1(\mathscr{E}^*) = 1, \qquad c_2(\mathscr{E}^*) = (1,0),$$
$$c_1(\mathscr{E}'^*) = 1, \qquad c_2(\mathscr{E}'^*) = (0,1).$$

Furthermore we see in the above sequences that both $\mathscr{E}^*$ and $\mathscr{E}'^*$ are generated by global sections. A section s of $\mathscr{E}'^*$, for instance has a Koszul sequence

$$0 \longrightarrow \mathcal{O}_G \longrightarrow \mathscr{E}'(1) \xrightarrow{s} \mathscr{J}_{Y'}(1) \longrightarrow 0$$

where Y' is an α'-plane and $\mathscr{J}_{Y'}$ is the ideal sheaf defining Y'. The cohomology dimensions $h^i(\mathscr{J}_{Y'}(1))$ are very easy to compute, so using this sequence we can easily obtain the following Table 1 for $h^i(\mathscr{E}(1)) = h^i(\mathscr{E}'(1))$.

Table 1. $h^i(\mathscr{E}(l)) = h^i(\mathscr{E}'(l))$

$i \backslash l$	-4	-3	-2	-1	0	1
4	4	0	0	0	0	0
3	0	0	0	0	0	0
2	0	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	0	0	4

In fact $H^0(\mathscr{E}(1)) = V$, $H^0(\mathscr{E}'(1)) = V^*$. We also observe that $\Lambda^2\mathscr{E}$ and $\Lambda^2\mathscr{E}'$ are isomorphic to each other, for both are isomorphic to $\mathcal{O}_G(-1)$. (However this isomorphism is not canonical and sometimes we will distinguish between these two sheaves.)

Theorem 1. *Given a coherent sheaf $\mathscr{F}$ on G, there is a spectral sequence $E_r^{p,q}$ with*

$$E_1^{-4,q} = H^q(\mathscr{F}(-2)) \otimes S^2\Lambda^2\mathscr{E} \qquad (\textit{recall } S^n\Lambda^2\mathscr{E} \simeq \mathcal{O}_G(-n)),$$
$$E_1^{-3,q} = H^q(\mathscr{F} \otimes \mathscr{E}'(-1)) \otimes \mathscr{E}(-1),$$
$$E_1^{-2,q} = (H^q(\mathscr{F}(-1)) \otimes S^2\mathscr{E}) \oplus (H^q(\mathscr{F} \otimes S^2\mathscr{E}') \otimes \Lambda^2\mathscr{E}),$$
$$E_1^{-1,q} = H^q(\mathscr{F} \otimes \mathscr{E}') \otimes \mathscr{E},$$
$$E_1^{0,q} = H^q(\mathscr{F}) \otimes \mathcal{O}_G,$$
$$E_1^{p,q} = 0 \quad \textit{for all other } p, q,$$

and with abutment E^n given by $E^0 = \mathscr{F}$, $E^n = 0$ for $n \neq 0$ and this is also the abutment if we interchange $\mathscr{E}$ and $\mathscr{E}'$.

Proof. The identity map in V corresponds through isomorphisms

$$\begin{aligned}\operatorname{Hom}(V,V) &\simeq V^*\otimes V \simeq H^0(\mathscr{E}'^*)\otimes H^0(\mathscr{E}^*)\\ &\simeq H^0(G\times G, \mathrm{pr}_1\,\mathscr{E}'^*\otimes \mathrm{pr}_2\,\mathscr{E}^*)\\ &\simeq H^0(G\times G, \mathscr{H}om(\mathrm{pr}_1^*\mathscr{E}', \mathrm{pr}_2^*\mathscr{E}^*))\end{aligned}$$

to a section s of $\mathscr{H}om(\mathrm{pr}_1^*\mathscr{E}', \mathrm{pr}_2^*\mathscr{E}^*)$ defined on each point $(\lambda_1,\lambda_2)\in G$, corresponding to lines $L_1=P(V_1^*)$, $L_2=P(V_2^*)$ by

$$s(\lambda_1,\lambda_2): V_1\ni\vec{v}\longrightarrow \vec{v}+V_2\in V/V_2.$$

This is the zero map if and only if $\lambda_1=\lambda_2$, i.e. the section s vanishes exactly on the diagonal $\Delta_{G\times G}$. From now on, tensor products of the kind $\mathrm{pr}_1^*\mathscr{F}_1\otimes\mathrm{pr}_2^*\mathscr{F}_2$ will be denoted by $\mathscr{F}_1\boxtimes\mathscr{F}_2$. Using this notation we write down the Koszul resolution

$$\begin{aligned}&0\longrightarrow \Lambda^4\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\Lambda^3\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\Lambda^2\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\\ &\longrightarrow\mathscr{O}_{G\times G}\longrightarrow\mathscr{O}_{\Delta_{G\times G}}\longrightarrow\mathscr{O}.\end{aligned}$$

This can also be written as

$$\begin{aligned}&0\longrightarrow S^2\Lambda^2\mathscr{E}'\boxtimes S^2\Lambda^2\mathscr{E}\longrightarrow\mathscr{E}'\otimes\Lambda^2\mathscr{E}'\boxtimes\mathscr{E}\otimes\Lambda^2\mathscr{E}\longrightarrow\\ &\longrightarrow(S^2\mathscr{E}\boxtimes\Lambda^2\mathscr{E}')\oplus(S^2\mathscr{E}'\boxtimes\Lambda^2\mathscr{E})\longrightarrow\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\mathscr{O}_{G\times G}\longrightarrow\mathscr{O}_{\Delta_{G\times G}}\longrightarrow 0.\end{aligned}$$

Since $\operatorname{Tor}_i^{G\times G}(\mathrm{pr}_1^*\mathscr{F},\mathscr{O}_{\Delta_{G\times G}})=0$, the sequence is still exact after tensoring by $\mathrm{pr}_1^*\mathscr{F}$. Omiting the last term we get the complex

$$\begin{aligned}&0\longrightarrow\mathscr{F}(-2)\boxtimes S^2\Lambda^2\mathscr{E}\longrightarrow\mathscr{F}(-1)\otimes\mathscr{E}'\boxtimes\mathscr{E}\otimes\Lambda^2\mathscr{E}\longrightarrow\\ &\longrightarrow(\mathscr{F}(-1)\boxtimes S^2\mathscr{E})\oplus(\mathscr{F}\otimes S^2\mathscr{E}\boxtimes\Lambda^2\mathscr{E})\longrightarrow\mathscr{F}\otimes\mathscr{E}'\boxtimes\mathscr{E}\longrightarrow\mathrm{pr}_1^*\mathscr{F}\longrightarrow 0.\end{aligned}$$

(Remark that if we choose $\mathrm{pr}_2\,\mathscr{F}$ instead of $\mathrm{pr}_1\,\mathscr{F}$ we end up with the analogous sequence, with $\mathscr{E},\mathscr{E}'$ interchanged.)

This is a complex

$$\mathscr{C}^\bullet:\quad 0\longrightarrow\mathscr{C}^{-4}\longrightarrow\mathscr{C}^{-3}\longrightarrow\mathscr{C}^{-2}\longrightarrow\mathscr{C}^{-1}\longrightarrow\mathscr{C}^{0}\longrightarrow 0$$

whose only nonzero cohomology group is $H^0(\mathscr{C}^\bullet)=\mathscr{F}|_{\Delta_{G\times G}}$.

We now consider the two spectral sequences abuting to the hypercohomology of this complex. The 'second' spectral sequence has

$$E_2^{p,q}=R^q\,\mathrm{pr}_{2*}(H^q\mathscr{C}^\bullet)=\begin{cases}\mathrm{pr}_{2*}(\mathrm{pr}_2^*\mathscr{E}\,|_\Delta)=\mathscr{F} & \text{if } p=q=0,\\ 0 & \text{otherwise.}\end{cases}$$

The abutment of this spectral sequence must be E^n with $E^0=\mathscr{F}$, $E^n=0$ for $n\neq 0$, and so must be the abutment of the 'first' spectral sequence whose terms ${}^{\mathrm{I}}E^{p,q}=R^q\,\mathrm{pr}_{2*}\,\mathscr{C}^p$ are easily computed by the projection formula [4, Ch II, Ex. 5.1.d], so that we obtain the list presented in the statement. This completes the proof of the theorem. $\square$

In practical uses, the dimensions of all cohomology spaces involved in this theorem are easy to compute (by tensoring F with the sequence (1) if necessary) the only exception being $h^q(\mathcal{F}\otimes S^2\mathcal{E}')$ which now receives our attention. We consider the two graphs, F, $\tilde{F}$ of the incidence relations obvious in the diagrams

$$\begin{tikzcd} & \mathbb{P}^3\times G \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\ & F \arrow[u, hook] \arrow[dl, "\mu"] \arrow[dr, "\nu"'] & \\ \mathbb{P}^3 & & G \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{P}^3\times G \arrow[ddl, "\tilde{p}_1"'] \arrow[ddr, "\tilde{p}_2"] & \\ & \tilde{F} \arrow[u, hook] \arrow[dl, "\tilde{\mu}"] \arrow[dr, "\tilde{\nu}"'] & \\ \mathbb{P}^{3*} & & G \end{tikzcd} \tag{1}$$

where $\mathbb{P}^{3*}$ stands for dual projective space. All of $\mathbb{P}^3\times G$, F, $\mathbb{P}^{3*}\times G$, $\tilde{F}$ have $\mathbb{Z}\otimes\mathbb{Z}$ as Picard group (inherited from both projections). For a coherent sheaf $\mathcal{S}$ on $\mathcal{O}_{\mathbb{P}^3\times G}$ we write $\mathcal{S}(m,n)$ instead of $\mathcal{S}\otimes p_1^*\mathcal{O}_{\mathbb{P}^3}(m)\otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(n)$ and we use similar conventions for the others.

Proposition 2. *In the case where the coherent sheaf $\mathcal{F}$ is locally free, $H^q(\mathcal{F}\otimes S^2\mathcal{E}')$ can be computed by means of two long exact sequences*

$$\cdots\longrightarrow H^{q-1}(\mathcal{F}\otimes S^2\mathcal{E}')\longrightarrow H^q(p_2^*\mathcal{F}\otimes\mathcal{J}(2,-2))\longrightarrow$$
$$\longrightarrow H^q(\mathcal{F}(-2)\otimes H^0(\mathcal{O}_{\mathbb{P}^3}(2)))\longrightarrow H^q(\mathcal{F}\otimes S^2\mathcal{E}')\longrightarrow\cdots$$

and

$$\cdots\longrightarrow H^{q-1}(p_2^*\mathcal{F}\otimes\mathcal{J}(2,-2))\longrightarrow H^q(\mathcal{F}(-3))\longrightarrow$$
$$\longrightarrow H^q(\mathcal{F}\otimes\mathcal{E}(-2))\otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1))\longrightarrow H^q(p_2^*\mathcal{F}\otimes\mathcal{J}(2,-2))\longrightarrow\cdots$$

where $\mathcal{J}$ is the ideal sheaf of F in $\mathbb{P}^3\times G$. In particular, if for all $q\ge 0$, $h^q(\mathcal{F}(-3))=h^q(\mathcal{F}(-2))=0$, then $h^q(\mathcal{F}\otimes S^2\mathcal{E}')=4h^{q+1}(\mathcal{F}\otimes\mathcal{E}(-2))$.

The same statement holds with $\mathcal{E}$, $\mathcal{E}'$ interchanged, and replacing $F,\mathbb{P}^3,p_2$ by $\tilde{F},\mathbb{P}^{3*},\tilde{p}_2$.

Proof. We use the fact that F considered as a $\mathbb{P}^1$-fibration (via ν) is nothing but $P(\mathcal{E}'^*)$, and $\mathcal{O}_{P(\mathcal{E}'^*)}(1)$ is $\mu^*\mathcal{O}_{\mathbb{P}^3}(1)=\mathcal{O}_F(1,0)$. Therefore (cf. [4, Chap. III, Ex. 8.4]) for $n\ge 0$

$$\nu_*\mathcal{O}_F(n,0)=S^n\mathcal{E}'^*\simeq S^n\mathcal{E}'(n),$$
$$R^q\nu_*\mathcal{O}_F(n,0)=0\quad\text{if } q>0.$$

Replacing F by $\tilde{F}$ we similarly obtain for $n\ge 0$

$$\tilde{\nu}_*\mathcal{O}_{\tilde{F}}(n,0)=S^n\mathcal{E}^*\simeq S^n\mathcal{E}(n),$$
$$R^q\tilde{\nu}_*\mathcal{O}_{\tilde{F}}(n,0)=0\quad\text{if } q>0.$$

Consider now the Leray spectral sequence

$$E_2^{q,p} = H^q(G, R^p v_* v^* \mathscr{F}(2,-2)) \Longrightarrow H^n(G, v^*\mathscr{F}(2,-2)).$$

As F is locally free, using the projection formula we compute

$$E_2^{q,0} = H^q(\mathscr{F} \otimes S^2\mathscr{E}'),$$
$$E_2^{q,p} = 0 \quad \text{for } p \neq 0,$$

so the spectral sequence degenerates and we obtain

$$H^q(F \otimes S^2\mathscr{E}') = H^q(v^*\mathscr{F}(2,-2)).$$

To go on with our computation we define a section s of

$$\mathscr{H}om(p_1^*\mathscr{O}_{\mathbb{P}^3}(-1), p_2^*E^*(1,0)).$$

For each point $(s,l) \in P^3 \times G$ with $x = P(V_1^*)$ and l corresponding to a line $L = P(V_2^*)$ define $s(x,l)$ as the map

$$s(x,l): V_1 \ni \vec{v} \longmapsto \vec{v} + V_2 \in V/V_2.$$

As F is the scheme of zeroes of section s we have an exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^3\times G}(-2,-1) \arrow[r] & p_2^*\mathscr{E}(-1,0) \arrow[rr] \arrow[dr] & & \mathscr{O}_{\mathbb{P}^3\times G} \arrow[r] & \mathscr{O}_F \arrow[r] & 0 \\
& & & \mathscr{J} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{2}$$

Since $\mathscr{F}$ is locally free, the sequences below are still exact

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & p_2^*\mathscr{F}(0,-3) \arrow[r] & p_2^*(\mathscr{F}\otimes\mathscr{E})(1,-2) \arrow[rr] \arrow[dr] & & p_2^*\mathscr{F}(2,-2) \arrow[r] & v^*\mathscr{F}(2,-2) \arrow[r] & 0 \\
& & & p_2\mathscr{F}\otimes\mathscr{J}(2,-2) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

From these two short sequences we will obtain two long sequences in cohomology and applying Küneth formula these two sequences become as stated. □

Proposition 3. *The bundle $S^2\mathscr{E}$ can be presented in the following way*

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & S^2\mathscr{E} \arrow[r] & \mathscr{E}\otimes V^* \arrow[rr] \arrow[dr] & & \mathscr{O}_G\otimes\Lambda^2V^* \arrow[r] & \mathscr{O}_G(1) \arrow[r] & 0 \\
& & & \Omega_{\mathbb{P}^5}(1)\otimes\mathscr{O}_G \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Analogously

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & S^2\mathcal{E}' \arrow[r] & \mathcal{E}'\otimes V \arrow[rr] \arrow[dr] & & \mathcal{O}_G\otimes\Lambda^2 V^* \arrow[r] & \mathcal{O}_G(1) \arrow[r] & 0 \\
& & & \Omega_{\mathbb{P}^5}(1)\otimes\mathcal{O}_G \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Proof. We apply Proposition 2 to the case where $\mathcal{F}=\mathcal{O}_G(1)$. In this case we have the following cohomology Tables 2 and 3.

Table 2. $h^q(\mathcal{F}(l))$

q \ l	-3	-2	-1	0
4	0	0	0	0
3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	0	0	1	6

Table 3. $h^q(\mathcal{F}\otimes\mathcal{E}'(l))$

q \ l	-3	-2	-1	0
4	0	0	0	0
3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	0	0	0	4

Thus $h^q v^* F(2,-2)=0$ for all $q\geq 0$ by Proposition 2.

In fact $H^0(F\otimes E')=V^*$ and $H^0(F)=\Lambda^2 V^*$, thus in the spectral sequence of Theorem 1 we have terms $E_1^{p,q}$ as expressed by Table 4.

Table 4. $E_1^{p,q}$

q \ p	-4	-3	-2	-1	0
4	0	0	0	0	0
3	0	0	0	0	0
2	0	0	0	0	0
1	0	0	0	0	0
0	0	0	$S^2\mathcal{E} \xrightarrow{\alpha}$	$\mathcal{E}\otimes V^* \xrightarrow{\beta}$	$\mathcal{O}_G\otimes\Lambda^2 V^*$

Looking at the abutment E^n of this spectral sequence (Theorem 1) we conclude that

$$\ker\alpha = 0, \qquad \ker\beta = \operatorname{im}\alpha, \qquad \operatorname{coker}\beta = \mathcal{O}_G(1).$$

We get $S^2\mathcal{E}$ presented by

$$0 \longrightarrow S^2\mathcal{E} \xrightarrow{\alpha} \mathcal{E}\otimes V^* \xrightarrow{\beta} \mathcal{O}_G\otimes\Lambda^2 V^* \longrightarrow \mathcal{O}_G(1) \longrightarrow 0$$

which splits in two short sequences as stated. □

A consequence of this proposition is that we can now handle $H^q(F\otimes S^2\mathcal{E}')$ for any coherent sheaf $\mathcal{F}$ on G, because

$$\mathrm{Tor}_1(\mathscr{F}, \Omega_{\mathbb{P}^5}(1)) = 0, \qquad \mathrm{Tor}_1(\mathscr{F}, \mathscr{O}_G(1)) = 0.$$

We can thus obtain from Proposition 3

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \otimes S^2\mathscr{E}' \arrow[r] & \mathscr{F} \otimes \mathscr{E}' \otimes V \arrow[rr] \arrow[dr] & & \mathscr{F} \otimes \Lambda^2 V^* \arrow[r] & \mathscr{F}(1) \arrow[r] & 0 \\
& & & \mathscr{F} \otimes \Omega_{\mathbb{P}^5}(1) \arrow[ur] & & & \\
& & 0 \arrow[ur] & & 0 \arrow[ul] & &
\end{tikzcd}$$

leading to the following

Corollary 4. *For any coherent sheaf $\mathscr{F}$ on G, $H^q(\mathscr{F} \otimes S^2\mathscr{E}')$ can be computed by two exact sequences*

$$\cdots \longrightarrow H^{q-1}(\mathscr{F} \otimes \Omega_{\mathbb{P}^5}(1)) \longrightarrow H^q(\mathscr{F} \otimes S^2\mathscr{E}') \longrightarrow H^q(\mathscr{F} \otimes \mathscr{E}') \otimes V \longrightarrow$$
$$\longrightarrow H^q(\mathscr{F} \otimes \Omega_{\mathbb{P}^5}(1) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H^{q-1}(\mathscr{F}(1)) \longrightarrow H^q(\mathscr{F} \otimes \Omega_{\mathbb{P}^5}(1)) \longrightarrow H^q(\mathscr{F}) \otimes \Lambda^2 V^* \longrightarrow$$
$$\longrightarrow H^q(\mathscr{F}(1) \longrightarrow \cdots$$

In particular, if $H^q(\mathscr{F}) = H^q(\mathscr{F}(1)) = 0$ for every integer q, then $H^q(\mathscr{F} \otimes S^2\mathscr{E}') \simeq H^q(\mathscr{F} \otimes \mathscr{E}') \otimes V$. The analogous holds for $H^2(\mathscr{F} \otimes S^2\mathscr{E})$.

As an example of an application, we now give a presentation of $S^n\mathscr{E}'$ in terms of bundles $\mathscr{O}_G(n), \mathscr{E}, S^2\mathscr{E}$.

Proposition 5. *There is an exact sequence for each $n \geq 2$*

$$0 \longrightarrow \binom{n+2}{3}\mathscr{E}(1) \longrightarrow \binom{n+3}{3} S^2\mathscr{E} \otimes 4\binom{n+2}{3}\mathscr{O}_G(-1) \longrightarrow 4\binom{n+3}{3}\mathscr{E} \longrightarrow$$
$$\longrightarrow \left[6\binom{n+3}{3} - 4\binom{n+2}{2} + \binom{n+1}{3}\right]\mathscr{O}_G \longrightarrow S^n\mathscr{E}'(n+1) \longrightarrow 0$$

and the analogous holds with $\mathscr{E}, \mathscr{E}'$ interchanged.

Proof. Let us apply Proposition 2 in the case where $\mathscr{F} = S^n\mathscr{E}'(n+1)$, with $n \geq 2$. For all integer $l \in \mathbb{Z}$

$$h^q(F(l)) = h^q(S^n(\mathscr{E}'(1))(1+l) = h^q(v_*\mathscr{O}_F(n, 1+l)) = h^q(\mathscr{O}_F(n, 1+l)).$$

(The last equality follows from the degeneracy of Leray's spectral sequence, as $R^q v_* \mathscr{O}_F(n, 1+l) = 0$ for all $j \geq 1$. We need to compute this dimension in cases $l = 0, -1, -2, -3$. Twisting sequence (2) we obtain

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3 \times G}(n-2, l) \arrow[r] & p_2^* \mathcal{E}(n+1, 1+l) \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^3 \times G}(n, 1+l) \arrow[r] & \mathcal{O}_F(n, 1+l) \arrow[r] & 0 \\
& & & \mathcal{J} \arrow[ur] \arrow[dl] \arrow[dr] & & & \\
& & 0 & & 0 & &
\end{tikzcd}$$

Using the Küneth formula we get $h^q(\mathcal{J}(n, 1+l)) = 0$ for all $q \geq 0$ and values of n, l that we are considering, with the exception $h^0(\mathcal{J}(n, 1)) = 4\binom{n+2}{3} - \binom{n+1}{3}$. Therefore $h^q(\mathcal{O}_F(n, 1+l)) = 0$ in these cases, with the exceptions

$$h^0(\mathcal{O}_F(n, 1)) = 6\binom{n+3}{3} - 4\binom{n+2}{3} + \binom{n+1}{3}, \qquad h^0(\mathcal{O}_F(n, 0)) = \binom{n+3}{3}.$$

According to Proposition 2 this implies that $h^q(S^2 \mathcal{E}' \otimes \mathcal{F}) = 4h^{q+1}(\mathcal{F} \otimes \mathcal{E}(-2))$. Now we compute $h^q(\mathcal{F} \otimes \mathcal{E}'(l))$ for $l = 0, -1$. This is $h^q(v^* \mathcal{E}'(n, 1+l)$ because Leray's spectral sequence still degenerates. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_G(-2) \longrightarrow \Omega_{\mathbb{P}^5} \otimes \mathcal{O}_G \longrightarrow \Omega_G \longrightarrow 0$$

and the fact that $\Omega_G = \mathcal{E} \otimes \mathcal{E}'$ we can easily compute dimensions (see Table 5).

Table 5. $h^q(E \otimes E'(t))$

$q \backslash t$	-2	-1	0	1
4	0	0	0	0
3	1	0	0	0
2	0	0	0	0
1	0	0	1	0
0	0	0	0	0

Using this information in the sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & p_2^* \mathcal{E}'(n-2, l) \arrow[r] & p_2^* \mathcal{E}' \otimes \mathcal{E}(n-1, 1+l) \arrow[rr] \arrow[dr] & & p_2^* \mathcal{E}'(n, 1+l) \arrow[r] & v^* \mathcal{E}'(n, 1+l) \arrow[r] & 0 \\
& & & \mathcal{J} \otimes p_2^* \mathcal{E}'(n, 1+l) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

we obtain estimations

$$h^q(\mathcal{J} \otimes p_2^* \mathcal{E}'(n+1)) = 0 \quad \text{for all } q,$$

$$h^q(\mathcal{J} \otimes p_2^* \mathcal{E}'(n, 0)) = 0 \quad \text{for all } q \text{ except } h^1(\mathcal{J} \otimes p_2^* \mathcal{E}'(n, 0)) = \binom{n+2}{3}.$$

Therefore

$$h^q(v^*\mathscr{E}'(n,1)) = 0 \quad \text{for all } q, \text{ except } h^0(v^*\mathscr{E}'(n,1)) = 4\binom{n+3}{3},$$

$$h^q(v^*\mathscr{E}'(n,0)) = 0 \quad \text{for all } q, \text{ except } h^0(v^*\mathscr{E}'(n,0)) = \binom{n+2}{3}.$$

Finally, we compute $h^q(\mathscr{F}\otimes\mathscr{E}(-2)) = h^q(v^*\mathscr{E}(n,-1))$. Using the sequence

$$0 \longrightarrow \mathscr{E}'\otimes\mathscr{E} \longrightarrow V\otimes\mathscr{E} \longrightarrow \mathscr{E}\otimes\mathscr{E}(1) \longrightarrow 0$$

and the data in Table 5 we obtain the following estimates (see Table 6).

Table 6. $h^q(\mathscr{E}\otimes\mathscr{E}(t))$

$q \backslash t$	-2	-1	0
4	0	0	0
3	1	0	0
2	0	1	0
1	0	0	0
0	0	0	0

We use this information in the sequence

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & p_2^*\mathscr{E}(n-2,2) \arrow[r] & p_2^*\mathscr{E}\otimes\mathscr{E}(n-1,1) \arrow[rr] \arrow[dr] & & p_2^*\mathscr{E}(n,-1) \arrow[r] & v^*\mathscr{E}(n,-1) \arrow[r] & 0 \\ & & & \mathscr{J}\otimes p_2^*\mathscr{E}(n-1,-1) \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

and obtain

$$h^q(\mathscr{J}\otimes p_2^*\mathscr{E}(n-1,-1)) = 0$$
$$\text{for all } q, \text{ except } h^2(\mathscr{J}\otimes p_2^*\mathscr{E}(n-1,-1)) = \binom{n+2}{3}.$$

Therefore,

$$h^q(v^*\mathscr{E}(n,-1)) = 0 \quad \text{for all } q, \text{ except } h^1(v^*\mathscr{E}(n,-1)) = \binom{n+2}{3}.$$

Now we know the terms $E_1^{p,q}$ of the spectral sequence of Theorem 1. These are all zero, except in the complex

$$E_1^{-3,0} \xrightarrow{\alpha} E_1^{-1,0} \xrightarrow{\beta} E_1^{-1,0} \longrightarrow E_1^{0,0}$$

that is

$$\binom{n+2}{3}\mathscr{E}(-1) \xrightarrow{\alpha} \binom{n+3}{3} S^2\mathscr{E} \otimes 4\binom{n+2}{3}\mathscr{O}_G(-1) \xrightarrow{\beta} 4\binom{n+3}{3}\mathscr{E} \longrightarrow$$

$$\xrightarrow{\gamma} \left[6\binom{n+3}{3} - 4\binom{n+2}{2} + \binom{n+1}{3}\right]\mathscr{O}_G.$$

Therefore all $E_\infty^{p,q}$ are zero, except

$$0 = E_\infty^{-3,0} = \ker \alpha, \qquad 0 = E_\infty^{-2,0} = \ker \beta/\operatorname{im} \alpha,$$
$$0 = E_\infty^{-1,0} = \ker \gamma/\operatorname{im} \beta, \qquad F = E_\infty^{0,0} = \operatorname{coker} \gamma$$

and Theorem 1 now yields the exact sequence of the statement. □

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